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NATURAL OPERATIONS ON COVARIANT TENSOR FIELDS

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1. Notation[1]

M will denote a connected n-dimensional C^∞-manifold, M_m the tangent space to M at m, and $(M_m)_s$ the space $\otimes^s M_m{}^*$. The differentiable covariant tensor fields of rank s on M form a real vector space V_s. Let $J(M)$ be the direct sum $V_1 \oplus \cdots \oplus V_r$. An automorphism ϕ of M induces an automorphism R_ϕ of V_s given by

$$R_\phi(v)_m(v_1, \cdots, v_s) = v_{\phi^{-1}(m)}(\delta\phi^{-1}(v_1) \cdots \delta\phi^{-1}(v_s)) .$$

If v is given in local coordinates (x) at $\phi^{-1}(m)$ by $v = \sum a(j_1, \cdots, j_s) \cdot dx(j_1) \cdots dx(j_s)$, then $R_\phi(v)$ near m by

$$(*) \qquad R_\phi(v) = \sum a(j_1, \cdots, j_s) \circ \phi^{-1} d(x(j_1) \circ \phi^{-1}) \cdots d(x(j_s) \circ \phi^{-1}) .$$

This representation $R: \phi \to R_\phi$ defines a representation of the group G of all automorphisms of M in V_s. If V and W are invariant subspaces, $J(V, W)$ will denote the space of intertwining operators:

$$J(V, W) = \{T \mid T: V \to W, T \text{ linear}, R_\phi \cdot T \cdot v = T \cdot R_\phi \cdot v \text{ for all } v \in V \text{ and all } \phi \in G\} .$$

G^0 is a certain subgroup of G introduced in [1, p. 127], and $J^0(V, W)$ is the corresponding space of intertwining operators. Clearly $J(V, W) \subseteq J^0(V, W)$. Let V_s be decomposed into irreducible invariant subspaces under G: $V_s = V_{s1} \oplus V_{s2} \oplus \cdots$. V_{si}, $i = 1, 2, \cdots$, are also invariant and irreducible under the group of permutations operating on the indices (symmetric classes).

2. Results

(1) $J^0(V_{si}, V_{rj}) = 0$ if $0 \le s < n, 0 < r \le n, r \ne s, r \ne s + 1$.
(2) $J^0(V_{sr}, V_{(s+1)t}) = 0$ if $r \ne 1$ or $t \ne 1$.
(3) $J^0(V_{s1}, V_{(s+1)1}) = J(V_{s1}, V_{(s+1)1}) =$ constant multiples of d^s.

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[1] Identical with that in Palais [1].

(4) $J^0(V_{sp}, V_{sq}) = 0$ if $p \neq q$.
(5) $J^0(V_{sm}, V_{sm})$ = algebra $A(s)$ of the permutation group of s indices.

Remark. (5) corresponds to the identity in [1], because the permutation group operates on differential forms as the identity multiplied by ± 1 according to the parity of the permutation.

3. Proof

Important facts in the proof of the results of Palais [1] are:

(a) the localizability of the elements of $J^0(V, W)$, [1, Theorem 6.1, p. 132],

(b) the decomposition of differential forms into a sum of differential forms of a very special kind at $m \in M$, which are called basic forms of the first or second kind at m, [1, p. 133],

(c) the construction of automorphisms out of germs of automorphisms, which have a very special form at m, adapted to the basic forms at m such that the results follow immediately from the naturality condition $R_\phi \cdot T \cdot v = T \cdot R_\phi \cdot v$ for a basic form v at m, [1, pp. 136–138].

Fortunately these arguments can be transfered to the case of covariant tensor fields. Without any assumption on the symmetry of v one can prove, by the naturality condition $R_\phi \cdot T \cdot v = T \cdot R_\phi \cdot v$, that the only natural linear operators T are the exterior derivative and the algebra of the permutation group. So v and Tv are either differential forms of rank s and $s+1$ respectively or belong to the same symmetric class V_{si}. One has to use the naturality condition for very special but rather simple automorphisms $\phi \in G^0$ to define basic tensor fields similar to the basic forms [1, p. 133] in the following way.

3.1. Definition. Let $w \in V_s$, $0 \leq s < n$. We say that w is a basic covariant s-tensor field of the first kind at $m \in M$, if there is a coordinate system (x) centered at m with spherical domain U such that the support of w is a compact subset of U and

$$w = f \cdot dx(i_1) \cdots dx(i_s) ,$$

where $f = x(i_{s+1})$, $(i_{s+1} \neq i_1, \cdots, i_{s+1} \neq i_s)$, for some neighborhood of m. We say that w is a basic s-tensor field of the first kind, if it is so at some $m \in M$.

3.2. Definition. Let $w \in V_s$, $0 \leq s \leq n$. We say that w is a basic s-tensor field of the second kind at $m \in M$, if there is a coordinate system (x) centered at m with spherical domain U such that the support of w is a compact subset of U and

$$w = f \cdot dx(i_1) \cdots dx(i_s) ,$$

where f is constant near m.

3.3. Theorem. *Let $T \in J^0(V_s, V_\tau)$, and w be a basic s-tensor field of the first kind at m, i.e., there is a coordinate system (x) centered at m with* *spherical domain U such that the support of w is a compact subset of U and*

$$w = x(i_{s+1}) \cdot dx(i_1) \cdots dx(i_s)$$

is some neighborhood V of m with $i_{s+1} \neq i_1, \cdots, i_{s+1} \neq i_s$. Then

$$Tw = \sum a(j_1, \cdots, j_r) \cdot (x(i_{s+1}))^{1-H(i_{s+1}|j_1,\cdots,j_r)} \cdot dx(j_1) \cdots dx(j_r)$$

near m with

$$(j_1, \cdots, j_r) = (i_1, \cdots, i_s, i_{s+1}, \cdots, i_{s+1}) ,$$

which means that the indices $j_1, \cdots, j_r$ are identical with the $i_1, \cdots, i_s, i_{s+1}, \cdots, i_{s+1}$ modulo permutations. $H(i_{s+1}|j_1, \cdots, j_r)$ is the number of times for i_{s+1} to occur in the indices $(j_1, \cdots, j_r)$, and $a(j_1, \cdots, j_r)$ are real constants. So the theorem says that all indices $i_1, \cdots, i_s$ occuring in the differentials of w must also occur in Tw and that i_{s+1} is the only index different from the $i_1, \cdots, i_s$, which can occur in Tw.

Proof. If $a(1), \cdots, a(n)$ are sufficiently small real numbers, then by [1, Lemma 5.4, p. 130] we can find a $\phi \in G^0$ such that $x(i) \cdot \phi^{-1} = x(i) + a(i)$ near m. Under the assumption

$$(1) \qquad Tw = \sum a(j_1, \cdots, j_r)(x) dx(j_1) \cdots dx(j_j)$$

formula $(*)$ in § 1 implies that

$$(2) \qquad R_\phi Tw = \sum a(j_1, \cdots, j_r) \circ \phi^{-1}(x) \cdot dx(j_1) \cdots dx(j_r) .$$

If $a(i_{s+1}) = 0$, then clearly $R_\phi w = w$ near m. Since T is localizable, and $TR_\phi w = Tw$ near m, from the relation $R_\phi Tw = TR_\phi w = Tw$ together with the expressions (1), (2) it follows that the $a(j_1, \cdots, j_r)(x)$ are function of $x(i_{s+1})$ alone:

$$Tw = \sum a(j_1, \cdots, j_r)(x(i_{s+1})) \cdot dx(j_1) \cdots dx(j_r) .$$

Given a real number c sufficiently close to unity, we can, appealing to [1, Lemma 5.4, p. 130], find a $\psi \in G^0$ such that we have, near m, $x(i) \cdot \psi^{-1} = x(i)$, $(i \neq i_{s+1})$, $x(i_{s+1}) \circ \psi^{-1} = cx(i_{s+1})$. Using the formula $(*)$ of § 1 and the fact that the $a(j_1, \cdots, j_r)$ depend only on $x(i_{s+1})$ near m, we have, near m,

$$(3) \quad R_\psi Tw = \sum a(j_1, \cdots, j_r)(c \cdot x(i_{s+1})) \cdot c^{H(i_{s+1}|j_1,\cdots,j_r)} \cdot dx(j_1) \cdots dx(j_r) .$$

On the other hand, it is clear that $R_\psi w = cw$ near m, so by the localizability and linearity of T we have at m:

$$(4) \quad R_\psi Tw = TR_\psi w = c \cdot Tw = c \cdot \sum a(j_1, \cdots, j_r)(x(i_{s+1})) dx(j_1) \cdots dx(j_r) .$$

Comparing (3) with (4) we have

$$a(j_1, \cdots, j_r)(x(i_{s+1})) = a(j_1, \cdots, j_r) \cdot x(i_{s+1})^{1-H(i_{s+1}|j_1,\cdots,j_r)} ,$$

where the $a(j_1, \cdots, j_r)$ are constants.

It is still to prove $(j_1, \cdots, j_r) = (i_1, \cdots i_s, i_{s+1}, \cdots, i_{s+1})$. For this purpose let $c(1), \cdots, c(n)$ be real numbers with $|c(i) - 1| < \varepsilon$, $c(i) \neq 1$ for $i \neq i_{s+1}$, and $c(i_{s+1}) = 1$. Let the $c(i)$, $(i \neq i_{s+1})$, satisfy the following condition: the relation

$$(5) \qquad c(i_1)^{m(1)} \cdots c(i_r)^{m(r)} = c(j_1)^{p(1)} \cdots c(j_t)^{p(t)} ,$$

with $i_1 < \cdots < i_r$, $j_1 < \cdots < j_t \in (i_1 \cdots i_s, \hat{i}_{s+1}, \cdots, i_n)$, and rational $m(i)$ and $p(i)$, is possible only if $r = t$, $i_1 = j_1, \cdots, i_r = j_r$ and $m(1) = p(1), \cdots, m(r) = p(r)$.

By [1, Lemma 5.4, p. 130] we can find a $\phi \in G^0$ such that $x(i) \circ \phi^{-1} = c(i) \cdot x(i)$ at m. Then at m

$$(6) \quad R_\phi Tw = \sum a(j_1, \cdots, j_r)(c(i_{s+1}) \cdot x(i_{s+1})) \cdot c(j_1) \cdots c(j_r) \cdot dx(j_1) \cdots dx(j_r) .$$

On the other hand,

$$R_\phi w = c(i_{s+1}) \cdot x(i_{s+1}) \cdot c(i_1) \cdots c(i_s) \cdot dx(i_1) \cdots dx(i_s) .$$

So again

$$(7) \quad \begin{aligned} R_\phi Tw = TR_\phi w = c(i_{s+1}) \cdot c(i_1) \cdots c(i_s) \\ \cdot \sum a(j_1, \cdots, j_r)(x(i_{s+1})) \cdot dx(j_1) \cdots dx(j_r) . \end{aligned}$$

Comparing (6) with (7) and using condition (5) for the $c(i)$ and $c(i_{s+1}) = 1$ we have

$$(j_1, \cdots, j_r) = (i_1, \cdots, i_s, i_{s+1}, \cdots, i_{s+1}) .$$

Remarks. (a) $Tw = 0$ near m for all $T \in J^0(V_s, V_r)$ with $s > r$ and for all basic fields w of the first kind at m.

(b) For $s = r$ or equivalently $H(i_{s+1}|j_1, \cdots, j_r) = 0$ we have

$$Tw = \sum a(j_1, \cdots, j_s) \cdot x(i_{s+1}) \cdot dx(j_1) \cdots dx(j_s) ,$$

which means that T operates on w as a symmetric operator $T \in A(s)$.

(c) For $r = s + 1$ or equivalently $H(i_{s+1}|j_1, \cdots, j_{s+1}) = 1$,

$$Tw = \sum a(j_1, \cdots, j_{s+1}) dx(j_1) \cdots dx(j_{s+1}) .$$

Since the proofs of the following Theorems 3.4 and 3.5 can follow along the lines very close to those of the proof of the preceding theorem, we mention only the necessary changes.

3.4. Theorem. *For $w \in V_s$ given at m by*

$$w = x(i_{s+1})\cdot x(i_p)^m \cdot dx(i_1) \cdots dx(i_s)$$

with $(i_{s+1} \neq i_1, \cdots, i_{s+1} \neq i_p)$ and for $T \in J^0(V_s, V_r)$ we have

$$(1) \qquad Tw = \sum b(j_1, \cdots, j_r)\cdot x(i_{s+1})^{1-H(i_{s+1}|j_1,\cdots,j_r)} \cdot x(i_p)^{m+H(i_p|i_1,\cdots,i_s)-H(i_p|j_1,\cdots,j_r)} \cdot dx(j_1) \cdots dx(j_r)$$

with $(i_1, \cdots, i_s, i_{s+1}, \cdots, i_{s+1}, i_p, \cdots, i_p) = (j_1, \cdots, j_r)$ and

$$r - s = H(i_p | j_1, \cdots, j_r) - H(i_p | i_1, \cdots, i_s) + H(i_{s+1} | j_1, \cdots, j_r) .$$

So only the occurence of i_{s+1} and i_p can change.

Proof. We assume near m

$$Tw = \sum b(j_1, \cdots, j_r)(x) dx(j_1) \cdots dx(j_r) .$$

Using the naturality condition with $\phi_1, \phi_2, \phi_3 \in G^0$ given at m by

$$x(i) \circ \phi_1^{-1} = x(i) + a(i) , \qquad a(i_{s+1}) = a(i_p) = 0 ,$$
$$x(i) \circ \phi_2^{-1} = x(i) , \quad i \neq i_{s+1} , \quad x(i_{s+1}) \circ \phi_2^{-1} = cx(i_{s+1}) ,$$
$$x(i) \circ \phi_3^{-1} = x(i) , \quad i \neq i_p , \quad x(i_p) \circ \phi_3^{-1} = cx(i_p) ,$$

we find the form of the functions $b(j_1, \cdots, j_r)(x)$. The indices are determined by a $\phi_4 \in G^0$ given by $x(i) \circ \phi_4^{-1} = c(i)\cdot x(i)$ according to (5) of § 3.3 with $c(i_{s+1}) = c(i_p) = 1$.

3.5. Corollary. *For $T \in J(V_s, V_{s+1})$ we have, near m,*

$$(1) \qquad Tw = x(i_p)^m \sum a(j_1, \cdots, j_{s+1})\cdot x(i_{s+1}) dx(j_1) \cdots dx(j_{s+1})$$

with $(j_1, \cdots, j_{s+1}) = (i_1, \cdots, i_s, i_{s+1})$.

$$v = x(i_{s+1})\cdot dx(i_1) \cdots dx(i_s) .$$

Proof. Let v be a basic field of the first kind at m:
According to [1, Lemma 5.1, p. 129] we have a $\phi \in G^0$ given by $x(i) \circ \phi^{-1} = x(i)$, $i \neq i_p$, $x(i_p) \circ \phi^{-1} = x(i_p) + x(i_p)^2$. Then from Remark (c) of § 3.3 and $m = H(i_p | i_1, \cdots, i_s)$ it follows that

$$(2) \qquad R_\phi T v = \sum [a(j_1, \cdots, j_{s+1}) + \textstyle\sum_{i=0,\cdots,m-1} a(j_1, \cdots, j_{s+1}) \cdot {}_m c_i (2x(i_p)^{m-i}] \cdot dx(j_1) \cdots dx(j_{s+1}) ,$$

$$(3) \qquad TR_\phi v = Tv + \textstyle\sum_{i=0,\cdots,m-1} {}_m c_i \cdot T(x(i_{s+1}) \cdot (2x(i_p))^{m-i} \cdot dx(i_1) \cdots dx(i_s)) .$$

The first terms on the right side of (2) and (3) cancel with each other. The second term of (3) is given by (1) of § 3.4. Since $H(i_p | i_1, \cdots, i_s) = H(i_p | j_1, \cdots, j_{s+1})$ holds in (2), it must also hold in (3) and (1) of § 3.4.

3.6. Theorem. *Let v be a basic field of the second kind at m:*

$$v = \text{const. } dx(i_1) \cdots dx(i_s) .$$

Then

(a) $T \cdot v = 0$ *for* $T \in J^0(V_s, V_r)$ *with* $r \neq s$,

(b) T *operates on* v *as symmetric operator*: $T \in A(s)$ *for* $T \in J(V_s, V_s)$.

Proof. Assume

$$T \cdot v = \sum f(j_1, \cdots, j_r)(x) \cdot dx(j_1) \cdots dx(j_r) .$$

By $\phi \in G^0$ given by $x(i) \circ \phi^{-1} = x(i) + a(i)$ we see that the $f(j_1, \cdots, j_r)(x)$ are constant. By $\phi \in G^0$ given by $x(i) \circ \phi^{-1} = c(i) \cdot x(i)$ according to (5) of § 3.3 with all $c(i) \neq 1$ we obtain

$$(j_1, \cdots, j_r) = (i_1, \cdots, i_s) \quad \text{or} \quad f(j_1, \cdots, j_r)(x) = 0 .$$

4. The cases $J^0(V_s, V_{s+m})$, $m = 0, 1, 2, \cdots$

According to Remark (a) of § 3.3 and Theorem 3.6 (a), basic fields of both kinds are mapped into zero by $T \in J^0(V_s, V_r)$ if $s > r$. So $J^0(V_s, V_r) = 0$, which proves partially (1) of § 2.

4.1. $J^0(V_s, V_s)$. According to Remark (b) of § 3.3 and Theorem 3.6 (b) every $T \in J^0(V_s, V_s)$ acts on both basic fields as a symmetric operator. From Palais [1, Lemma 7.3, p. 134, and the arguments of Theorem 10.3, p. 138], it follows that for both kinds of decomposition of a tensor field, T is the same symmetric operator. Thus by [1, Corollary 6.3, p. 133] the operator acts independently of $m \in M$. Hence (4) and (5) of § 2 are proved.

4.2. $J^0(V_s, V_{s+1})$. For $T \in J^0(V_s, V_{s+1})$ and a basic field of the first kind given by

$$v = x(i_{s+1}) \cdot dx(i_1) \cdots dx(i_s)$$

we have

$$\begin{aligned} T \cdot v &= \sum a(j_1, \cdots, j_{s+1}) dx(j_1) \wedge \cdots \wedge dx(j_{s+1}) \\ &= \text{const. } dx(j_{s+1}) \wedge dx(j_1) \wedge \cdots \wedge dx(j_s) . \end{aligned}$$

Proof. By $\phi \in G^0$ given by

$$x(i) \circ \phi^{-1} = x(i) , \quad i \neq i_{s+1} ; \quad x(i_{s+1}) \circ \phi^{-1} = x(i_{s+1}) + x(i_{s+1}) \cdot x(i_s) ,$$

and using the expression for Tv in Remark (c) of § 3.3 we obtain

$$(1)\qquad \begin{aligned} R_\phi Tw = &\sum a(j_1, \cdots, j_{s+1})\cdot dx(j_1) \cdots dx(j_{s+1}) \\ &+ x(i_s)\cdot \sum a(j_1, \cdots, j_{s+1}) dx(j_1) \cdots dx(j_{s+1}) \\ &+ x(i_{s+1}) \sum a(k_1, \cdots, k_{s+1}) dx(k_1) \cdots dx(k_{s+1}) \end{aligned}$$

with $(j_1, \cdots, j_{s+1}) = (i_1, \cdots, i_s, i_{s+1})$, $(k_1, \cdots, k_{s+1}) = (i_1, \cdots, i_s, i_s)$,

$$(2)\qquad TR_\phi w = Tw + T(x(i_{s+1})\cdot x(i_s)\cdot dx(i_1) \cdots dx(i_s)) .$$

The first terms on the right side of (1) and (2) cancel with each other. For the second term on the right side of (2) we have (1) of § 3.5 with $m = 1$, which now becomes

$$(3)\qquad x(i_{s+1})\cdot \sum a(k_1, \cdots, k_{s+1}) dx(k_1) \cdots dx(k_{s+1}) = 0$$

with $(k_1, \cdots, k_{s+1}) = (i_1, \cdots, i_s, i_s)$. (Notice the two equal indices i_s.) As the same expression (3) but with different indices i_s, i_{s+1} is not zero (Remark (c), § 3.3), the antisymmetry in k_s, k_{s+1} of (3) is proved. By repeating these arguments for every index $i_r \in (i_1, \cdots, i_s)$, we thus obtain the antisymmetry between k_{s+1} and every k_r in (3), which means the total antisymmetry in all $k_1, \cdots, k_{s+1}$. By linearity of $T \in J^0(V_s, V_r)$, tensor fields, which are not totally antisymmetric, are mapped into zero by T. So the arguments of Palais [1, Theorem 10.5, p. 139] give that $J^0(V_s, V_{s+1}) =$ constant d^s.

4.3. $J^0(V_s, V_{s+m})$, $m > 1$. According to Theorem 3.6 (a) basic fields of the second kind are mapped into zero. For basic fields of the first kind, by Theorem 3.3 we have

$$Tw = \sum a(j_1, \cdots, j_{s+m})\cdot (x(i_{s+1}))^{1-m} dx(j_1) \cdots dx(j_{s+m}) .$$

Using (1) of § 3.4 and the naturality condition with $\phi \in G^0$ given by

$$x(i) \circ \phi^{-1} = x(i) , \quad i \neq i_{s+1} , \quad x(i_{s+1}) \circ \phi^{-1} = x(i_{s+1})(1 + x(i_s))$$

we obtain $Tw = 0$, which together with § 4 proves (1) of § 2.

Reference

[1] R. S. Palais, *Natural operations on differential forms,* Trans. Amer. Math. Soc. **92** (1959 125–138.

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